

Oscillation of a Linear Delay Impulsive Differential Equation

L. Berezhansky, ¹

Ben-Gurion University of the Negev,
Department of Mathematics and Computer Science,
Beer-Sheva 84105, Israel,

E. Braverman

Technion - Israel Institute of Technology,
Israel Institute of Metals, 32000, Haifa, Israel

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Abstract

The main result of the paper is that the oscillation (non-oscillation) of the impulsive delay differential equation

$$\dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] = 0, \quad t \geq 0,$$

$$x(\tau_j) = B_j x(\tau_j - 0), \quad \lim \tau_j = \infty$$

is equivalent to the oscillation (non-oscillation) of the equation without impulses

$$\dot{x}(t) = \sum_{k=1}^m A_k(t) \prod_{h_k(t) < \tau_j \leq t} B_j^{-1} x[h_k(t)] = 0, \quad t \geq 0.$$

Explicit oscillation results are presented.

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1 Introduction

Recently results on oscillation of delay differential equations have taken shape of a developed theory presented in monographs [1-4]. At the same time it is an intensively developing field which is an objective of numerous publications.

However, for impulsive differential equations there are only few publications dealing with oscillation problems [1,4,5,6].

The purpose of the present paper is to fill up this gap. The main result is that the oscillation (non-oscillation) of the impulsive delay differential equation is equivalent to the oscillation (non-oscillation) of a certain differential equation without impulses which can be constructed explicitly from an impulsive equation. Thus the oscillation problems (in particular, oscillation and non-oscillation criteria) for an impulsive equation can be reduced to the similar problem for a certain non-impulsive equation.

The method proposed in the present paper for oscillation is new both for impulsive and non-impulsive equations. It is based on the solution representation formula. Recently such formulas are widely used in stability investigations of non-impulsive [7-9] and impulsive equations [5,10-12].

We demonstrate that the existence of a nonoscillating solution is equivalent to the positiveness of the fundamental function. At the same time this is equivalent to the solvability of a certain nonlinear inequality which is similar to "the generalized characteristic equation" from the monograph [2].

The paper is organized as follows. Theorems 1 and 2 are concerned with the equivalence of non-oscillation, positiveness of a fundamental function and solvability of a certain inequality. They lead to explicit non-oscillation results (Theorem 3). Theorem 4 compares non-oscillation conditions for two different impulsive delay differential equations. Theorems 5 and 6 give new oscillation criteria for delay differential equations without impulses. Theorem 7 contains the main result of the paper connecting oscillation of an impulsive and a non-impulsive equation. As a corollary (Theorem 8) we obtain explicit oscillation conditions for an impulsive delay equation.

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2 Preliminaries

We consider a scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] = f(t), \quad t \geq 0; \quad (1)$$

$$x(\tau_j) = B_j x(\tau_j - 0), \quad j = 1, 2, \dots, \quad (2)$$

under the following assumptions

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ are fixed points, $\lim_{j \rightarrow \infty} \tau_j = \infty$;

(a2) $A_k, f, k = 1, \dots, m$ are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $B_j \in \mathbf{R}$, $j = 1, \dots$, \mathbf{R} is a real axis;

(a3) $h_k : [0, \infty) \rightarrow \mathbf{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$.

Together with (1),(2) we will consider for each $t_0 \geq 0$ an initial value problem

$$\dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] = f(t), \quad \text{where } t \geq t_0, \quad x(\xi) = \varphi(\xi), \xi < t_0, \quad (3)$$

$$x(\tau_j) = B_j x(\tau_j - 0), \quad \tau_j > t_0. \quad (4)$$

We assume that for the initial function φ the following hypothesis holds

(a4) $\varphi : (-\infty, t_0) \rightarrow \mathbf{R}$ is a Borel measurable bounded function.

Definition. An absolutely continuous on each interval $[\tau_j, \tau_{j+1})$ function $x : [t_0, \infty) \rightarrow \mathbf{R}$ is a solution of the impulsive problem (3),(4) if (3) is satisfied for almost all $t \in [0, \infty)$ and the equalities (4) hold.

Definition. For each $s \geq 0$ the solution $X(t, s)$ of the problem

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] &= 0, \quad \text{where } t \geq s; \quad x(\xi) = 0, \quad \xi < s; \\ x(\tau_j) &= B_j x(\tau_j - 0), \quad \tau_j > s, \quad x(s) = 1, \end{aligned} \quad (5)$$

is a fundamental function of the equation (1),(2).

We assume $X(t, s) = 0$, $0 \leq t < s$.

Lemma 1 [12] *Let (a1)-(a4) hold. Then there exist one and only one solution of the problem (3) with the initial condition $x(t_0) = \alpha_0$ and impulsive conditions*

$$x(\tau_j) = B_j x(\tau_j) + \alpha_j$$

that can be presented in the form

$$x(t) = X(t, t_0)x(t_0) + \int_{t_0}^t X(t, s)f(s)ds - \sum_{k=1}^m \int_{t_0}^t X(t, s)A_k(s)\varphi[h_k(s)]ds + \sum_{\tau_j > t_0} X(t, \tau_j)\alpha_j, \quad (6)$$

where $\varphi[h_k(s)] = 0$, if $h_k(s) > t_0$.

3 Non-oscillation Criteria for Impulsive Equations

Definition. The equation (1),(2) has a *non-oscillating solution* if there exist $t_0 > 0$, $\varphi(t)$ satisfying (a4) such that for $f \equiv 0$ the solution of (3),(4) is positive for $t \geq t_0$. Otherwise, all solutions of (1),(2) are said to be *oscillating*.

In sequel we accept that the following hypothesis holds

(a5) delays are bounded: for every $s > 0$

$$\mu_s = \min_k \inf_{t \geq s} h_k(t) > -\infty$$

and there exists $s' \geq s$ such that $h_k(t) \geq s$ if $t \geq s'$.

Denote for any s

$$\begin{aligned} A_k^s(t) &= \begin{cases} A_k(t), & \text{if } t \geq s, \\ 0, & \text{if } t < s, \end{cases} \\ h_k^s(t) &= \begin{cases} h_k(t), & \text{if } t \geq s, \\ s, & \text{if } t < s. \end{cases} \end{aligned} \quad (7)$$

The following theorem establishes non-oscillation criteria.

Theorem 1 Suppose (a1)-(a5) hold, $A_k(t) \geq 0$, $k = 1, \dots, m$, and $B_j > 0$, $j = 1, 2, \dots$. Then the following hypotheses are equivalent

- 1) The equation (1),(2) has a non-oscillating solution.
- 2) There exists $t_0 \geq 0$ such that $X(t, s) > 0$, $t_0 \leq s < t < \infty$.
- 3) For a certain $t_1 \geq 0$ there exists a non-negative integrable in each interval $[t_1, b]$ solution u of an inequality

$$u(t) \geq \sum_{k=1}^m A_k^{t_1}(t) \exp \left\{ \int_{h_k^{t_1}(t)}^t u(s)ds \right\} \prod_{h_k^{t_1}(t) < \tau_j \leq t} B_j^{-1}, \quad t \geq t_1. \quad (8)$$

Here and in sequel we assume that a product equals to unit if number of factors is equal to zero.

Proof. The scheme of the proof is $1) \implies 3) \implies 2) \implies 1)$.

1) \implies 3). Let $x(t)$ be a positive solution of (3),(4) ($f \equiv 0$). By (a5) for a certain $t_1 \geq t_0$ $h_k(t) > t_0$, $t \geq t_1$, $k = 1, \dots, m$.

Let us demonstrate that

$$u(t) = -\frac{d}{dt} \ln \left\{ \frac{x(t)}{x(t_1)} \prod_{t_1 < \tau_j \leq t} B_j^{-1} \right\}, \quad t \geq t_1.$$

is a solution of (8). To this end we integrate the latter equality

$$x(t) = x(t_1) \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j, \quad t \geq t_1. \quad (9)$$

By setting $\varphi(t) = x(t)$ for $t < t_1$ one obtains that $x(t)$, $t \geq t_1$, is a solution of (3),(4), with the initial point $t = t_1$ and the initial function $\varphi(t) > 0$. We substitute (9) in (3) ($f \equiv 0$):

$$\begin{aligned} & -u(t) \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j + \\ & \sum_{k \in N_1} A_k(t) \exp \left\{ - \int_{t_1}^{h_k(t)} u(s) ds \right\} \prod_{t_1 < \tau_j \leq h_k(t)} B_j + \\ & \sum_{k \in N_2} A_k(t) \varphi[h_k(t)] = 0, \quad t \geq t_1. \end{aligned} \quad (10)$$

Here $N_1 = \{k : h_k(t) \geq t_1\}$, $N_2 = \{k : h_k(t) < t_1\}$.

Using notations (7) the equality (10) can be rewritten in the form

$$\begin{aligned} & -u(t) \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j + \\ & \sum_{k=1}^m A_k^{t_1}(t) \exp \left\{ - \int_{t_1}^{h_k^{t_1}(t)} u(s) ds \right\} \prod_{t_1 < \tau_j \leq h_k^{t_1}(t)} B_j + \end{aligned}$$

$$\sum_{k \in N_2} A_k(t) \varphi[h_k(t)] = 0, \quad t \geq t_1.$$

Consequently,

$$\begin{aligned} & \left(u(t) - \sum_{k=1}^m A_k^{t_1}(t) \exp \left\{ \int_{h_k^{t_1}(t)}^t u(s) ds \right\} \prod_{h_k^{t_1}(t) < \tau_j \leq t} B_j^{-1} \right) \times \\ & \times \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j = \sum_{k \in N_2} A_k(t) \varphi[h_k(t)] \geq 0, \end{aligned}$$

since $\varphi(t)$ is positive according to our choice of the point t_1 , which implies 3).

3) \implies 2). Consider (3),(4) with the initial function $\varphi \equiv 0$ and initial value $x(t_1) = 0$ in a segment $[t_1, b]$:

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t) x[h_k(t)] &= f(t), \quad t \in [t_1, b] : \quad x(\xi) = 0, \xi < t_1, \\ x(t_1) &= 0, \quad x(\tau_j) = B_j x(\tau_j - 0), \quad \tau_j > t_1. \end{aligned} \quad (11)$$

Besides, we consider an ordinary impulsive differential equation including the solution $u(t) \geq 0$ of (8):

$$\begin{aligned} \dot{x}(t) + u(t)x(t) &= z(t), \quad t \in [t_1, b], \\ x(\tau_j) &= B_j x(\tau_j - 0), \quad x(t_1) = 0. \end{aligned} \quad (12)$$

The solution of (12) can be rewritten in the form [15]

$$x(t) = \int_{t_1}^t \exp \left\{ - \int_s^t u(\xi) d\xi \right\} \prod_{s < \tau_j \leq t} B_j z(s) ds. \quad (13)$$

We seek for the solution of (11) of the form (13). By substituting x and \dot{x} from (13) and (12) into (11), we obtain

$$\begin{aligned} & z(t) - u(t) \int_{t_1}^t \exp \left\{ - \int_s^t u(\xi) d\xi \right\} \prod_{s < \tau_j \leq t} B_j z(s) ds + \\ & \sum_{k=1}^m A_k^{t_1}(t) \int_{t_1}^{h_k^{t_1}(t)} \exp \left\{ - \int_s^{h_k^{t_1}(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h_k^{t_1}(t)} B_j z(s) ds = f(t). \end{aligned} \quad (14)$$

The equation (14) is of the type

$$z - Hz = f, \quad (15)$$

where

$$\begin{aligned} (Hz)(t) = & u(t) \int_{t_1}^t \exp \left\{ - \int_s^t u(\xi) d\xi \right\} \prod_{s < \tau_j \leq t} B_j z(s) ds - \\ & \sum_{k=1}^m A_k^{t_1}(t) \int_{t_1}^{h_k^{t_1}(t)} \exp \left\{ - \int_s^{h_k^{t_1}(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h_k^{t_1}(t)} B_j z(s) ds. \end{aligned} \quad (16)$$

It is well known [14] that the integral operator

$$(Hz)(t) = \int_{t_1}^b K(t, s) z(s) ds$$

acting in the space $\mathbf{L}_{[t_1, b]}$ of functions integrable on $[t_1, b]$ is compact if

$$|K(t, s)| \leq k(t), \quad k \in \mathbf{L}_{[t_1, b]}. \quad (17)$$

For the operator H defined by (16)

$$|K(t, s)| \leq \sup_{s, t \in [t_1, b]} \prod_{s < \tau_j \leq t} B_j \left(u(t) + \sum_{k=1}^m |A_k(t)| \right).$$

Thus the inequality (17) holds and the operator $H : \mathbf{L}_{[t_1, b]} \rightarrow \mathbf{L}_{[t_1, b]}$ is a compact Volterra integral operator. Therefore [14] its spectral radius is equal to zero. Consequently the equation (15) for any $f \in \mathbf{L}_{[t_1, b]}$ has a single solution

$$z = (I - H)^{-1} f, \quad (18)$$

where I is the identity operator.

Let us show that H is a positive operator. The operator H can be easily rewritten as a sum $H = H_1 + H_2$, where

$$\begin{aligned} (H_1 z)(t) = & \left(u(t) - \sum_{k=1}^m A_k^{t_1}(t) \exp \left\{ \int_{h_k^{t_1}(t)}^t u(s) ds \right\} \prod_{h_k^{t_1}(t) < \tau_j \leq t} B_j^{-1} \right) \times \\ & \times \int_{t_1}^t \exp \left\{ - \int_s^t u(\xi) d\xi \right\} \prod_{s < \tau_j \leq t} B_j z(s) ds, \end{aligned}$$

$$(H_2 z)(t) = \sum_{k=1}^m A_k^{t_1}(t) \int_{h_k^{t_1}(t)}^t \exp \left\{ - \int_s^{h_k^{t_1}(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h_k^{t_1}(t)} B_j z(s) ds.$$

The inequality (8) implies $H_1 \geq 0$. So $H = H_1 + H_2 \geq 0$. Since the spectral radius of H is equal to zero, then

$$(I - H)^{-1} = I + H + H^2 + \dots \geq 0.$$

Thus if $f \geq 0$, then the solution z of (15) is non-negative: $z \geq 0$.

The solution of (11) has the form (13), where z is the solution of (15). Consequently, if in (11) $f \geq 0$, then for the solution of (11) $x \geq 0$.

On the other hand, the solution of (11) can be presented in the form (6)

$$x(t) = \int_{t_1}^t X(t, s) f(s) ds. \quad (19)$$

As shown above, $f \geq 0$ implies $x \geq 0$. Therefore the kernel of the integral operator is non-negative, i.e. $X(t, s) \geq 0$ for $t_1 \leq s \leq t < b$. Since $b > t_1$ is chosen arbitrarily, then $X(t, s) \geq 0$ for $t_1 \leq s < t < \infty$.

Let us prove that in fact the strict inequality $X(t, s) > 0$ holds.

Denote

$$x(t) = X(t, t_1) - \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j.$$

Our purpose is to demonstrate $x(t)$ is non-negative. The function $x(t)$ is a solution of (3),(4), with $x(t_1) = 0, \varphi \equiv 0$ and

$$\begin{aligned} f(t) &= u(t) \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j - \\ &- \sum_{k=1}^m A_k^{t_1}(t) \exp \left\{ - \int_{t_1}^{h_k^{t_1}(t)} u(s) ds \right\} \prod_{t_1 < \tau_j \leq h_k^{t_1}(t)} B_j = \\ &= \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j \times \\ &\times \left(u(t) - \sum_{k=1}^m A_k^{t_1}(t) \exp \left\{ \int_{h_k^{t_1}(t)}^t u(s) ds \right\} \prod_{h_k^{t_1}(t) < \tau_j \leq t} B_j^{-1} \right). \end{aligned}$$

Thus (8) implies $f(t) \geq 0$. Therefore in view of (6)

$$x(t) = \int_{t_1}^t X(t, s)f(s)ds \geq 0.$$

Consequently,

$$X(t, t_1) \geq \exp \left\{ - \int_{t_1}^t u(s)ds \right\} \prod_{t_1 < \tau_j \leq t} B_j > 0.$$

For $s > t_1$ the inequality $X(t, s) > 0$ can be proven similarly.

2) \implies 1). Denote $x(t) = X(t, t_0)$. Then $x(t)$ is a positive solution of (3),(4) ($f \equiv 0$) with the initial function $\varphi \equiv 0$. The proof is complete.

Let us consider (1),(2) with coefficients of an arbitrary sign.

Denote $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$.

Theorem 2 Suppose (a1)-(a5) hold and $B_j > 0$.

Consider three hypotheses:

1) The initial value problem (3),(4) with an initial point $t_0 > 0$ ($f \equiv 0$) has a positive solution that continuously extend the continuous initial function φ .

2) $X(t, s) > 0$, $t_0 \leq s < t < \infty$.

3) There exists a non-negative integrable on each interval $[t_0, b]$ solution of an inequality

$$u(t) \geq \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \exp \left\{ \int_{h_k^{t_0}(t)}^t u(s)ds \right\} \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1}, \quad t \geq t_0. \quad (20)$$

Then implications $3) \implies 2)$, $3) \implies 1)$ are valid.

Proof. The proof of $3) \implies 2)$ coincides with the proof of $3) \implies 2)$ in Theorem 1 up to the place where the operator H is presented as a sum of two terms. Here

$$H = H_1 + H_2 + H_3,$$

where

$$(H_1 z)(t) = \left(u(t) - \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \exp \left\{ \int_{h_k^{t_0}(t)}^t u(s)ds \right\} \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1} \right) \times$$

$$\begin{aligned}
& \times \int_{t_0}^t \exp \left\{ - \int_s^t u(\xi) d\xi \right\} \prod_{s < \tau_j \leq t} B_j z(s) ds, \\
(H_2 z)(t) &= \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \int_{h_k^{t_0}(t)}^t \exp \left\{ - \int_s^{h_k^{t_0}(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h_k^{t_0}(t)} B_j z(s) ds, \\
(H_3 z)(t) &= \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^- \int_{t_0}^{h_k^{t_0}(t)} \exp \left\{ - \int_s^{h_k^{t_0}(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h_k^{t_0}(t)} B_j z(s) ds.
\end{aligned}$$

Again, like in Theorem 1, $H_1 \geq 0$, $H_2 \geq 0$, $H_3 \geq 0$, which implies $H = H_1 + H_2 + H_3 \geq 0$. The end of the proof completely repeats the corresponding one of Theorem 1.

3) \implies 1). Let us consider the problem (3),(4). Let μ_{t_0} be chosen as in the hypothesis (a5). We extend to the interval $[\mu_{t_0}, t_0)$ the coefficients $A_k(t)$ by zero and the delays $h_k(t)$ such that $h_k(t) \leq t$. Let $u(t)$ be a non-negative function satisfying (20). We extend it by zero to $[\mu_{t_0}, t_0)$. Then $u(t)$ is a solution of (20), where t_0 is changed by μ_{t_0} .

Consider a corresponding extension of (3),(4) to the interval $[\mu_{t_0}, \infty)$. As proven above, 3) \implies 2), therefore $X(t, s) > 0$ for $\mu_{t_0} \leq s < t < \infty$. Assuming

$$\begin{aligned}
\varphi(t) &= X(t, \mu_{t_0}) \text{ for } \mu_{t_0} \leq t < t_0 \quad \text{and} \\
x(t) &= X(t, \mu_{t_0}) \text{ for } t \geq t_0,
\end{aligned}$$

we obtain that $x(t)$ is a positive solution of (3),(4) ($f \equiv 0$), with an initial point t_0 , that continuously extends the continuous initial function φ . This completes the proof of the theorem.

Now we proceed to explicit non-oscillation results.

Denote

$$\underline{h}^{t_0}(t) = \min_k h_k^{t_0}(t),$$

where $h_k^s(t)$ is defined by (7).

Theorem 3 *Suppose (a1)-(a5) hold, $B_j > 0$ and at least one of the following three hypotheses hold:*

1) $A_k(t) \leq 0$, $t \geq t_0$.

$$2) \quad \text{vrai} \sup_{t > t_0} \sum_{k=1}^m \int_{\underline{h}^{t_0}(t)}^t \left(A_k^{t_0}(s) \right)^+ \prod_{h_k^{t_0}(s) < \tau_j \leq s} B_j^{-1} z(s) ds \leq 1/e. \quad (21)$$

$$3) \sum_{k=1}^m \int_{\underline{h}^{t_0}(t)}^t \left(A_k^{t_0}(s)\right)^+ ds \leq 1/e \left(1 + \sum_{h_k^{t_0}(t) < \tau_j \leq t, B_j < 1} \ln B_j\right), \quad t \geq t_0. \quad (22)$$

Then the fundamental matrix $X(t, s)$ is positive for $t_0 \leq s < t < \infty$ and there exists a positive solution of (3),(4) ($f \equiv 0$) continuously extending a continuous initial function φ .

Proof. Obviously 1) is a special case of 2). Let us prove the theorem assuming (21) holds. To this end we will demonstrate that a function

$$u(t) = e \sum_{k=1}^m \left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1}$$

is a non-negative solution of the inequality (20). By substituting u in (20) one obtains

$$\begin{aligned} & e \sum_{k=1}^m \left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1} \geq \sum_{k=1}^m \left(A_k^{t_0}(t)\right)^+ \times \\ & \times \exp \left\{ e \int_{h_k^{t_0}(t)}^t \sum_{i=1}^m \left(A_i^{t_0}(s)\right)^+ \prod_{h_i^{t_0}(s) < \tau_j \leq s} B_j^{-1} ds \right\} \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1}. \end{aligned}$$

This unequality can be deduced from the following one

$$\begin{aligned} & e \sum_{k=1}^m \left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1} \geq \\ & \exp \left\{ e \int_{\underline{h}^{t_0}(t)}^t \sum_{k=1}^m \left(A_k^{t_0}(s)\right)^+ \prod_{h_k^{t_0}(s) < \tau_j \leq s} B_j^{-1} ds \right\} \times \\ & \times \sum_{k=1}^m \left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1}. \end{aligned}$$

After dividing this inequality by its left-hand side and logarithmizing it we obtain

$$\sum_{k=1}^m \int_{\underline{h}^{t_0}(t)}^t \left(A_k^{t_0}(s)\right)^+ \prod_{h_k^{t_0}(s) < \tau_j \leq s} B_j^{-1} z(s) ds \leq 1/e,$$

which obviously results from (21).

Let 3) hold. We will prove that

$$u(t) = e \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+$$

is a solution of the inequality (20) which after substituting takes form

$$e \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \geq \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \exp \left\{ e \int_{h_k^{t_0}(t)}^t \sum_{i=1}^m \left(A_i^{t_0}(s) \right)^+ ds \right\} \prod_{h_k^{t_0}(t) < \tau_j \leq t} B_j^{-1}.$$

This inequality can be deduced from

$$e \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \geq \sum_{k=1}^m \left(A_k^{t_0}(t) \right)^+ \exp \left\{ e \int_{\underline{h}^{t_0}(t)}^t \sum_{k=1}^m \left(A_k^{t_0}(s) \right)^+ ds \right\} \prod_{\underline{h}^{t_0}(t) < \tau_j \leq t} B_j^{-1},$$

where the product contains only factors for which $B_j < 1$. The latter inequality after dividing by the left-hand side and logarithmizing coincides with (22). This completes the proof of the theorem.

Let us compare oscillation properties of (1),(2) and an impulsive equation

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m \tilde{A}_k(t)x(\tilde{h}_k(t)) &= f(t), \quad t \in [0, \infty), \\ x(\tau_j) &= \tilde{B}_j x(\tau_j - 0). \end{aligned} \tag{23}$$

Theorem 4 *Let the hypotheses (a1)-(a5) hold for the equations (1),(2) and (23), $\tilde{A}_k(t) \geq 0, B_j > 0$. Suppose that any (therefore, all) of the hypotheses 1)-3) of Theorem 1 holds for (1),(2).*

Then if $A_k(t) \geq \tilde{A}_k(t)$, $B_j \leq \tilde{B}_j$ and at least one of the hypotheses

1) $h_k(t) \leq \tilde{h}_k(t)$, $\tilde{B}_j \leq 1$, $j = 1, 2, \dots$;

2) $h_k(t) = \tilde{h}_k(t)$,

holds then for the equation (23) the assertions 1)-3) of Theorem 1 are valid.

Proof. By the hypothesis of the theorem there exists a non-negative function $u(t)$ satisfying (7). Besides, for any non-negative function u under the hypotheses of the theorem the inequality

$$\begin{aligned} & \sum_{k=1}^m A_k(t) \exp \left\{ \int_{h_k(t)}^t u(s) ds \right\} \prod_{h_k(t) < \tau_j \leq t} B_j^{-1} \geq \\ & \geq \sum_{k=1}^m \tilde{A}_k(t) \exp \left\{ \int_{\tilde{h}_k(t)}^t u(s) ds \right\} \prod_{\tilde{h}_k(t) < \tau_j \leq t} \tilde{B}_j^{-1} \end{aligned}$$

holds. Consequently if u is a solution of the inequality (7) then u is a solution of this inequality, where A_k, h_k, B_j are changed by $\tilde{A}_k, \tilde{h}_k, \tilde{B}_j$. Then by Theorem 1 the other assertions of this theorem also hold.

Corollary 1. Suppose the hypotheses (a1)-(a5) hold for (1),(2) and $B_j > 0$. Besides, let $0 \leq A_k(t) \leq A_k$, $t - h_k(t) \leq h_k$, $B_j \leq 1$.

If there exists a non-oscillating solution of the equation with constant coefficients and delays

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k x(t - h_k) &= f(t), \quad t \in [0, \infty), \\ x(\tau_j) &= B_j x(\tau_j - 0), \end{aligned}$$

then there exists a non-oscillating solution of the equation (1),(2).

Corollary 2. Let (a1)-(a5) hold and $A_k(t) \geq 0$. If there exists a non-oscillation solution of the equation (1) without impulses and $B_j \geq 1$, then there exists a non-oscillating solution of the impulsive equation (1),(2).

4 Oscillation Properties of Impulsive and Non-impulsive Equations

Consider a non-impulsive differential equation

$$\dot{x}(t) + \sum_{k=1}^m a_k(t) x[h_k(t)] = f(t), \quad t \geq 0. \quad (24)$$

Denote by $x(t, s)$ the fundamental function of the equation (24). After substituting $B_j \equiv 1$ Theorems 1 and 2 immediately yield the following results.

Theorem 5 Suppose (a2)-(a5) hold for (24) and $a_k(t) \geq 0, k = 1, 2, \dots$. Then the following hypotheses are equivalent:

- 1) The equation (24) has a non-oscillating solution ($f \equiv 0$).
- 2) There exists $t_0 \geq 0$ such that $x(t, s) > 0$ for $t_0 \leq s < t < \infty$.
- 3) For a certain $t_1 \geq 0$ there exists a non-negative integrable on each interval $[t_1, b]$ solution u of the inequality

$$u(t) \geq \sum_{k=1}^m a_k^{t_1}(t) \exp \left\{ \int_{h_k^{t_1}(t)}^t u(\xi) d\xi \right\}, \quad t \geq t_1. \quad (25)$$

Theorem 6 Suppose (a2)-(a5) hold for (24). Consider three hypotheses:

- 1) The initial value problem for (24) ($f \equiv 0$) with an initial point $t_0 \geq 0$ has a positive solution that is a continuous expansion of a continuous initial function φ ;
- 2) $x(t, s) > 0, t_0 \leq s < t < \infty$;
- 3) There exists a non-negative integrable on each interval $[t_0, b]$ solution u of the inequality

$$u(t) \geq \sum_{k=1}^m \left(a_k^{t_0}(t) \right)^+ \exp \left\{ \int_{h_k^{t_0}(t)}^t u(s) ds \right\}, \quad t \geq t_0$$

Then implications $3) \Rightarrow 2), 3) \Rightarrow 1)$ are valid.

Corollary of Theorem 3 for the equation (24) coincides with the known non-oscillation result for equations without impulses [1,2,4].

In this paper we present a fundamental result that enables to reduce the oscillation problem for (1),(2) to the oscillation problem for an equation without impulses. To this end consider an auxiliary equation

$$\dot{x}(t) + \sum_{k=1}^m A_k(t) \prod_{h_k^0(t) < \tau_j \leq t} B_j^{-1} x[h_k(t)] = 0, \quad t \in [0, \infty), \quad (26)$$

$$\text{where} \quad h_k^0(t) = \begin{cases} h_k(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Denote by $Y(t, s)$ a fundamental function of the equation (26).

Theorem 7 Suppose (a1)-(a5) hold, $A_k \geq 0$, $B_j > 0$.

Then

1) There exists $t_0 > 0$, such that $X(t, s) > 0$, $t_0 \leq s < t < \infty \iff$ there exists $t_1 > 0$, such that $Y(t, s) > 0$, $t_1 \leq s < t < \infty$.

2) All solutions of (1),(2) ($f \equiv 0$) are oscillating \iff all solutions of (26) are oscillating.

3) There exists a non-oscillating solution of (1),(2) ($f \equiv 0$) \iff there exists a non-oscillating solution of (26).

Proof. 1). Let $X(t, s) > 0$, $t_0 \leq s < t < \infty$. Then by Theorem 1 there exists a solution of the inequality (7) for $t \geq t_1$. This inequality coincides with (25) under

$$a_k(t) = A_k(t) \prod_{h_k^0(t) < \tau_j \leq t} B_j^{-1}.$$

Therefore by Theorem 5 $Y(t, s) > 0$, $t_1 \leq s < t < \infty$. The converse can be proven similarly.

2). Suppose all solutions of (1),(2) ($f \equiv 0$) are oscillating and (26) has a positive solution, beginning with a certain t_0 . Then by Theorem 5 $Y(t, s) > 0$ for $t_1 \leq s < t < \infty$. Then, as proven in 1), $X(t, s) > 0$ for $t_2 \leq s < t < \infty$. Consequently, by Theorem 1 the equation (1),(2) has a non-oscillating solution, which contradicts to the hypothesis. The converse is proven similarly.

Besides, 2) implies 3), which completes the proof.

By applying Theorem 7 and known oscillation (non-oscillation) results on equations without impulses, one obtains oscillation results for impulsive equations. As an example we present the following statement.

$$\text{Denote } \underline{h}(t) = \min_k h_k(t), \bar{h}(t) = \max_k h_k(t).$$

Theorem 8 Let (a1)-(a5) hold for (1),(2), $A_k(t) \geq 0$ and $B_j > 0$. Then if at least one of the following inequalities holds

$$1) \liminf_{t \rightarrow \infty} \int_{\underline{h}(t)}^t \sum_{k=1}^m A_k(s) \prod_{h_k(s) < \tau_j \leq s} B_j^{-1} ds > 1/e,$$

$$2) \limsup_{t \rightarrow \infty} \int_{\bar{h}(t)}^t \sum_{k=1}^m A_k(s) \prod_{h_k(s) < \tau_j \leq s} B_j^{-1} ds > 1,$$

then all the solutions of (1),(2) are oscillating.

This statement is obtained by applying Theorem 7 and oscillation results for equations without impulses from the monographs [1,2,4].

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